VISCOELASTIC LAYER UNDER THE ACTION OF A MOVING LOAD

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Recently, problems concerning the dynamic behavior of imperfect continuous media under various types of actions have been widely investigated. The method of Laplace transformation is very convenient for describing physical processes concerning unsteady phenomena. In viscoelastic media two complications are added: the representation of the properties of a medium depending on time, and the inversion of the obtained solutions containing this additional complication. Certain approximate methods of inversion in the analysis of viscoelastic stresses are discussed in [1]. In [2, 3] a discussion is given for an effective method of constructing the solution of unsteady problems for finite and for infinite imperfect media using auxiliary functions, and a solution is presented for a half-space. Making use of the idea of the inversion of transforms, discussed in [4], in [5] a solution is obtained and a complete picture is presented for the dynamics of the variation of the stress field in a viscoelastic half-space. In the present study we consider the action of a normal moving load that is suddenly applied to the free surface of a viscoelastic layer. By Laplace and Fourier integral transformations we obtain a solution in the form of a uniformly converging series based on longitudinal and transverse waves reflected in the layer. By means of inverting the transforms by the method discussed in [4, 5], we obtain an exact solution for the stress field in the medium under investigation. We consider the special case of a viscoelastic medium of Boltzmann type, for which we obtain a numerical realization of the solution on a digital computer.

We are given a layer of thickness h of a viscoelastic material of Boltzmann type, covering the half-space $z > h$. The layer is rigidly fastened to the half-space. At time $t = 0$, to the surface of the layer $z = 0$ there is applied a load P_0, distributed along the y axis and moving along the x axis with constant velocity c_0. The problem of finding the stress field in a viscoelastic layer reduces to integration of the equations

$$\begin{cases} \dfrac{\partial\sigma_{xx}}{\partial x} + \dfrac{\partial\tau_{xz}}{\partial z} = \rho\dfrac{\partial^2 u}{\partial t^2}, \\ \dfrac{\partial\tau_{zx}}{\partial x} + \dfrac{\partial\sigma_{zz}}{\partial z} = \rho\dfrac{\partial^2 w}{\partial t^2}, \end{cases} \tag{1}$$

where u and w are the displacement components along the x and z axes, respectively, and

$$\sigma_{xx} = \int_0^t \left\{[\lambda\delta(t-\tau) - Q_1(t-\tau)]\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + 2[\mu\delta(t-\tau) - Q_2(t-\tau)]\frac{\partial u}{\partial x}\right\}d\tau; \tag{2}$$

$$\sigma_{zz} = \int_0^t \left\{[\lambda\delta(t-\tau) - Q_1(t-\tau)]\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + 2[\mu\delta(t-\tau) - Q_2(t-\tau)]\frac{\partial w}{\partial z}\right\}d\tau;$$

$$\tau_{xz} = \int_0^t [\mu\delta(t-\tau) - Q_2(t-\tau)]\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)d\tau,$$

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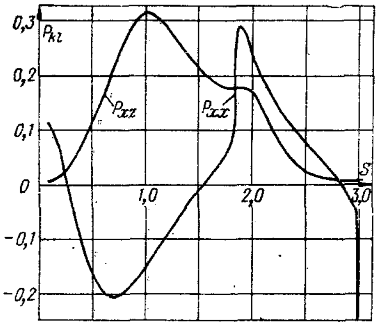

Fig. 1

where $Q_1(t)$ is the kernel of the volume relaxation; $Q_2(t)$ is the kernel of the shear relaxation; λ and μ are the Lamé constants; ρ is the density of the medium; and $\delta(t)$ is a delta function. Equations (1) are solved for the boundary conditions

$$\left\{\begin{array}{l}\sigma_{zz}=-P_0\delta(tc_0-x),\\ \tau_{xz}=0,\end{array}\right|_{z=0}, \tag{3}$$

where z = h is the rigid coupling of the media. The initial conditions are homogeneous.

We introduce into consideration the potential of longitudinal waves φ and the potential of transverse waves ψ. The solution of the formulated problem is found by the method of Laplace integral transformation with respect to time t, and bilateral complex Fourier transformation with respect to the variable x:

$$\Phi(\alpha, s; z)=Ae^{-\gamma_1 z}+Be^{\gamma_1 z};$$
$$\Psi(\alpha, s; z)=Ce^{-\gamma_2 z}+De^{\gamma_2 z}$$

(for the notation, see [5]). For the purpose of studying the successive reflection of waves from the boundaries of the layer, the potentials of the longitudinal and transverse waves are represented in the form [6]

$$\begin{aligned}\Phi(\alpha, s; z)&=\sum_{m,n=0}^{\infty} a_{mn}^{+}e^{-f_1^{+}(\alpha, s; z)}+\sum_{m,n=0}^{\infty} a_{mn}^{-}e^{-f_1^{-}(\alpha, s; z)};\\ \Psi(\alpha, s; z)&=\sum_{m,n=0}^{\infty} b_{mn}^{+}e^{-f_2^{+}(\alpha, s; z)}+\sum_{m,n=0}^{\infty} b_{mn}^{-}e^{-f_2^{-}(\alpha, s; z)},\end{aligned} \tag{4}$$

where

$$\begin{aligned}f_1^{+}&=mh\gamma_1+nh\gamma_2-\gamma_1 z-i\alpha x;\\ f_1^{-}&=(m-1)h\gamma_1+nh\gamma_2+\gamma_1 z-i\alpha x;\\ f_2^{+}&=mh\gamma_1+nh\gamma_2-\gamma_2 z-i\alpha x;\\ f_2^{-}&=mh\gamma_1+(n-1)h\gamma_2+\gamma_2 z-i\alpha x.\end{aligned}$$

For the coefficients $a_{mn}^{\pm}$ and $b_{mn}^{\pm}$ we can obtain recursion relations, using the boundary conditions (3), Eq. (2), and the representation of the potentials in the form (4).

For convenience in the investigations, the stress field $\bar{\sigma}_{kl}$ found in the transform space is divided into the parts $\bar{\sigma}_{kl}^{-}$ and $\bar{\sigma}_{kl}^{+}$ (k = x, z; l = x, z), due to waves propagating from the free boundary of the layer to the boundary of the media and from the boundary of the media to the free surface of the layer, respectively:

$$\bar{\sigma}_{kl}^{\pm}=\frac{P_0 v_0}{2\pi}\int_{-\infty}^{+\infty}\sum_{j=1}^{2}\frac{\bar{P}_{kl}^{(\pm j)}e^{-f_j^{\pm}(\alpha, s; z)}}{(i\alpha-v_0 s)}d\alpha,$$

where

$$\bar{P}_{kl}^{(\pm j)}=\sum_{m,n=0}^{\infty}\bar{P}_{klmn}^{(\pm j)}, \quad (m+n\neq 0).$$

For the terms $\bar{P}_{kl\,mn}^{(\pm j)}$ we obtain recursion relations; the initial terms $P_{kl}^{(-j)}$ are presented in [5]. Integrals along the real α axis of the complex plane are replaced by approximate contour integrals. A substitution of variables is made so that by integrating with respect to the corresponding variable, we obtain the stresses due to the potentials of the longitudinal and transverse waves:

$$\alpha=\alpha_j=s\bar{v}_j a_j p_j, \ (j=1, 2),$$

where

$$a_1 = \left(\frac{\lambda + 2\mu}{\rho}\right)^{\frac{1}{2}}; \quad a_2 = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}.$$

The integration is carried out assuming that [3]

$$\frac{\bar{v}_1(s)}{\bar{v}_2(s)} = \frac{a_2}{a_1} = \text{const}.$$

Following [4], we introduce a new variable τ, having dimensions of time and allowing us to carry out the integration only from the moment of arrival at the point being considered of the front of the corresponding wave

$$f_j^{\pm}(\alpha, s; z) = \tau, \quad (j = 1, 2). \tag{5}$$

The velocity of the moving load is assumed to be subseismic, and the contour of integration is deformed so that it does not enclose poles and branch points, which allows us to eliminate residues from the solution.

After transformation in the contour integrals to the variable τ and use of the general formula of inversion of Laplace transforms, the stress field is represented in the form

$$\sigma_{kl}^{\pm}(r, \theta; t) = \frac{P_0 v_0}{2\pi}\left\{\sum_{j=1}^{2}\int_0^{\infty} H\left(\tau - \tau_j^{\pm}\right)\frac{P_{kl}^{(\pm j)}}{R_j}\frac{\partial p_j^{\pm}}{\partial \tau}F_j(p_j, \tau, t)\Bigg|_{p_j^-}^{p_j^+} d\tau + \right.$$

$$+ H(\cos\theta - \nu)\int_0^{\infty} H\left(\tau_2^{\pm} - \tau\right) H\left(\tau - \tau^{\pm}\right)\frac{P_{kl}^{(\pm 2)}}{R_2}\frac{\partial p^{\pm}}{\partial \tau}F(p, \tau, t)\Bigg|_{p_-^+}^{p_+^+} d\tau +$$

$$\left. + H(-\cos\theta - \nu)\int_0^{\infty} H\left(\tau_2^{\pm} - \tau\right) H\left(\tau - \tau^{\pm}\right)\frac{P_{kl}^{(\pm 2)}}{R_2}\frac{\partial p^{\pm}}{\partial \tau} F(p, \tau, t)\Bigg|_{p_-^-}^{p_+^-} d\tau\right\},$$

where H(t) is the Heaviside function, $\nu = a_2/a_1$, $\tan\theta = z/x$, and

$$F_j(p_j, \tau, t) = L^{-1}\left\{\frac{e^{-s\bar{v}_j a_j \tau}}{ip_j + \frac{v_0}{a_j \bar{v}_j}}\right\}; \quad F(p, \tau, t) = L^{-1}\left\{\frac{e^{-s\bar{v}_2 a_2 \tau}}{ip + \frac{v_0}{a_2 \bar{v}_2}}\right\}.$$

The derivatives $\partial p_j^{\pm}/\partial\tau$ and $\partial p^{\pm}/\partial\tau$, and the limits $\tau_j^{\pm}$ and $\tau^{\pm}$ of the variation τ for the contours of integration are determined after solution of Eq. (5) for the new variable p_j.

Determining the functions F as a function of specific relaxation kernels, the stress field in the layer is found by the summation

$$\sigma_{kl} = \sigma_{kl}^{-} + \sigma_{kl}^{+}.$$

To numerically realize the obtained solution we choose the following relaxation functions:

$$Q_2(t) = Ae^{-\frac{\beta t}{\tau_0}}\frac{1}{T^{1-\alpha}};$$

$$Q_1(t) = \frac{\lambda}{\mu}Q_2(t).$$

The calculations are carried out in the dimensionless parameters

$$H = \frac{h}{a_2\tau_0}; \; P_{kl}^{\pm} = \frac{a_2\tau_0}{P_0}\sigma_{kl}^{\pm}; \; T = \frac{t}{\tau_0}; \; \eta_0 = \frac{\mu_0}{\mu}; \; \varkappa_0 = \frac{a_2}{c_0}; \; \varkappa = \frac{a_2}{a_1}$$

with account of the threefold passage of longitudinal and transverse waves through the layer.

Figure 1 shows the stresses P_{xx} and P_{xz} for $\eta_0 = 0.5$;

$$\varkappa_0 = 2; \ \varkappa = \frac{2}{3}; \ T = 2; \ H = 3; \ \theta = \frac{\pi}{2}.$$

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